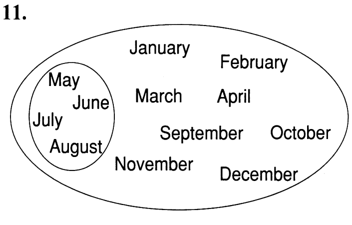
19. $\forall x \exists y \exists z (y \neq z \wedge \forall w (P(w, x) \leftrightarrow (w = y \vee w = z)))$ **21. a)** $\neg \exists x P(x)$ **b)** $\exists x (P(x) \wedge \forall y (P(y) \rightarrow y = x))$ **c)** $\exists x_1 \exists x_2 (P(x_1) \wedge P(x_2) \wedge x_1 \neq x_2 \wedge \forall y (P(y) \rightarrow (y = x_1 \vee y = x_2)))$ **d)** $\exists x_1 \exists x_2 \exists x_3 (P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge \forall y (P(y) \rightarrow (y = x_1 \vee y = x_2 \vee y = x_3)))$ **23.** Suppose that $\exists x (P(x) \rightarrow Q(x))$ is true. Then either $Q(x_0)$ is true for some x_0, in which case $\forall x P(x) \rightarrow \exists x Q(x)$ is true; or $P(x_0)$ is false for some x_0, in which case $\forall x P(x) \rightarrow \exists x Q(x)$ is true. Conversely, suppose that $\exists x (P(x) \rightarrow Q(x))$ is false. That means that $\forall x (P(x) \wedge \neg Q(x))$ is true, which implies $\forall x P(x)$ and $\forall x (\neg Q(x))$. This latter proposition is equivalent to $\neg \exists x Q(x)$. Thus, $\forall x P(x) \rightarrow \exists x Q(x)$ is false. **25.** No **27.** $\forall x \forall z \exists y T(x, y, z)$, where $T(x, y, z)$ is the statement that student x has taken class y in department z, where the domains are the set of students in the class, the set of courses at this university, and the set of departments in the school of mathematical sciences **29.** $\exists! x \exists! y T(x, y)$ and $\exists x \forall z ((\exists y \forall w (T(z, w) \leftrightarrow w = y)) \leftrightarrow z = x)$, where $T(x, y)$ means that student x has taken class y and the domain is all students in this class **31.** $P(a) \rightarrow Q(a)$ and $Q(a) \rightarrow R(a)$ by universal instantiation; then $\neg Q(a)$ by modus tollens and $\neg P(a)$ by modus tollens **33.** We give a proof by contraposition and show that if $\sqrt{x}$ is rational, then x is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational (q^2 is again nonzero). **35.** We can give a constructive proof by letting $m = 10^{500} + 1$. Then $m^2 = (10^{500} + 1)^2 > (10^{500})^2 = 10^{1000}$. **37.** 23 cannot be written as the sum of eight cubes. **39.** 223 cannot be written as the sum of 36 fifth powers.

CHAPTER 2

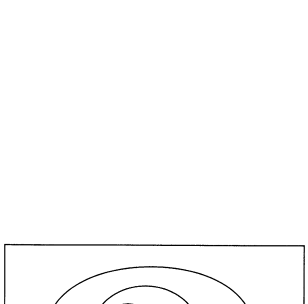
Section 2.1

1. a) $\{-1, 1\}$ **b)** $\{1,2,3,4,5,6,7,8,9,10,11\}$ **c)** $\{0,1,4,9,16,25,36,49,64,81\}$ **d)** $\emptyset$ **3. a)** Yes **b)** No **c)** No **5. a)** Yes **b)** No **c)** Yes **d)** No **e)** No **f)** No **7. a)** False **b)** False **c)** False **d)** True **e)** False **f)** False **g)** True **9. a)** True **b)** True **c)** False **d)** True **e)** True **f)** False

11.



13. The dots in certain regions indicate that those regions are not empty.



15. Suppose that $x \in A$. Because $A \subseteq B$, this implies that $x \in B$. Because $B \subseteq C$, we see that $x \in C$. Because $x \in A$ implies that $x \in C$, it follows that $A \subseteq C$. **17. a)** 1 **b)** 1 **c)** 2 **d)** 3 **19. a)** $\{\emptyset, \{a\}\}$ **b)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **c)** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ **21. a)** 8 **b)** 16 **c)** 2 **23. a)** $\{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$ **b)** $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$ **25.** The set of triples (a, b, c), where a is an airline and b and c are cities **27.** $\emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\} = \emptyset = \{(x, y) \mid x \in A \text{ and } y \in \emptyset\} = A \times \emptyset$ **29.** mn **31.** The elements of $A \times B \times C$ consist of 3-tuples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$, whereas the elements of $(A \times B) \times C$ look like $((a, b), c)$—ordered pairs, the first coordinate of which is again an ordered pair. **33. a)** The square of a real number is never -1. True **b)** There exists an integer whose square is 2. False **c)** The square of every integer is positive. False **d)** There is a real number equal to its own square. True **35. a)** $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0, 1\}$ **c)** $\emptyset$ **37.** We must show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. The "if" part is immediate. So assume these two sets are equal. First, consider the case when $a \neq b$. Then $\{\{a\}, \{a, b\}\}$ contains exactly two elements, one of which contains one element. Thus, $\{\{c\}, \{c, d\}\}$ must have the same property, so $c \neq d$ and $\{c\}$ is the element containing exactly one element. Hence, $\{a\} = \{c\}$, which implies that $a = c$. Also, the two-element sets $\{a, b\}$ and $\{c, d\}$ must be equal. Because $a = c$ and $a \neq b$, it follows that $b = d$. Second, suppose that $a = b$. Then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$, a set with one element. Hence, $\{\{c\}, \{c, d\}\}$ has only one element, which can happen only when $c = d$, and the set is $\{\{c\}\}$. It then follows that $a = c$ and $b = d$. **39.** Let $S = \{a_1, a_2, \ldots, a_n\}$. Represent each subset of S with a bit string of length n, where the ith bit is 1 if and only if $a_i \in S$. To generate all subsets of S, list all 2^n bit strings of length n (for instance, in increasing order), and write down the corresponding subsets.

Section 2.2

1. a) The set of students who live within one mile of school and who walk to classes **b)** The set of students who live within one mile of school or who walk to classes (or who do both) **c)** The set of students who live within one mile of school but do not walk to classes **d)** The set of students who walk to classes but live more than one mile away from school **3. a)** $\{0,1,2,3,4,5,6\}$ **b)** $\{3\}$ **c)** $\{1, 2, 4, 5\}$ **d)** $\{0, 6\}$ **5.** $\overline{\overline{A}} = \{x \mid \neg(x \in \overline{A})\} = \{x \mid \neg(\neg x \in A)\} = \{x \mid x \in A\} = A$ **7. a)** $A \cup U = \{x \mid x \in A \vee x \in U\} = \{x \mid x \in A \vee \mathbf{T}\} = \{x \mid \mathbf{T}\} = U$ **b)** $A \cap \emptyset = \{x \mid x \in A \wedge x \in \emptyset\} = \{x \mid x \in$

$A \wedge \mathbf{F}\} = \{x \mid \mathbf{F}\} = \emptyset$ **9. a)** $A \cup \overline{A} = \{x \mid x \in A \vee x \notin A\} = U$ **b)** $A \cap \overline{A} = \{x \mid x \in A \wedge x \notin A\} = \emptyset$ **11. a)** $A \cup B = \{x \mid x \in A \vee x \in B\} = \{x \mid x \in B \vee x \in A\} = B \cup A$ **b)** $A \cap B = \{x \mid x \in A \wedge x \in B\} = \{x \mid x \in B \wedge x \in A\} = B \cap A$ **13.** Suppose $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$ by the definition of intersection. Because $x \in A$, we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup B$ as well. Therefore $x \in A \cap (A \cup B)$ by the definition of intersection, so the right-hand side is a subset of the left-hand side. **15. a)** $x \in \overline{(A \cup B)} \equiv x \notin (A \cup B) \equiv \neg(x \in A \vee x \in B) \equiv \neg(x \in A) \wedge \neg(x \in B) \equiv x \notin A \wedge x \notin B \equiv x \in \overline{A} \wedge x \in \overline{B} \equiv x \in \overline{A} \cap \overline{B}$

b)

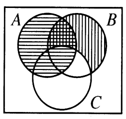
A	B	$A \cup B$	$\overline{(A \cup B)}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

17. a) $x \in \overline{A \cap B \cap C} \equiv x \notin A \cap B \cap C \equiv x \notin A \vee x \notin B \vee x \notin C \equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}$

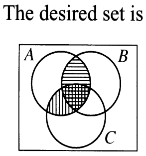
b)

A	B	C	$A \cap B \cap C$	$\overline{(A \cap B \cap C)}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

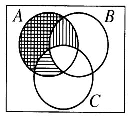
19. Both sides equal $\{x \mid x \in A \wedge x \notin B\}$. **21.** $x \in A \cup (B \cup C) \equiv (x \in A) \vee (x \in (B \cup C)) \equiv (x \in A) \vee (x \in B \vee x \in C) \equiv (x \in A \vee x \in B) \vee (x \in C) \equiv x \in (A \cup B) \cup C$ **23.** $x \in A \cup (B \cap C) \equiv (x \in A) \vee (x \in (B \cap C)) \equiv (x \in A) \vee (x \in B \wedge x \in C) \equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \equiv x \in (A \cup B) \cap (A \cup C)$ **25. a)** $\{4, 6\}$ **b)** $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ **c)** $\{4, 5, 6, 8, 10\}$ **d)** $\{0, 2, 4, 5, 6, 7, 8, 9, 10\}$ **27. a)** The double-shaded portion is the desired set.



b) The desired set is the entire shaded portion.



c) The desired set is the entire shaded portion.

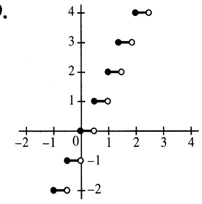


29. a) $B \subseteq A$ **b)** $A \subseteq B$ **c)** $A \cap B = \emptyset$ **d)** Nothing, because this is always true **e)** $A = B$ **31.** $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B) \equiv \forall x(x \notin B \rightarrow x \notin A) \equiv \forall x(x \in \overline{B} \rightarrow x \in \overline{A}) \equiv \overline{B} \subseteq \overline{A}$ **33.** The set of students who are computer science majors but not mathematics majors or who are mathematics majors but not computer science majors **35.** An element is in $(A \cup B) - (A \cap B)$ if it is in the union of A and B but not in the intersection of A and B, which means that it is in either A or B but not in both A and B. This is exactly what it means for an element to belong to $A \oplus B$. **37. a)** $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$ **b)** $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$ **c)** $A \oplus U = (A - U) \cup (U - A) = \emptyset \cup \overline{A} = \overline{A}$ **d)** $A \oplus \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = U$ **39.** $B = \emptyset$ **41.** Yes **43.** Yes **45. a)** $\{1, 2, 3, \ldots, n\}$ **b)** $\{1\}$ **47. a)** A_n **b)** $\{0, 1\}$ **49. a)** $\mathbf{Z}$, $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0\}$, $\emptyset$ **c)** $\mathbf{R}$, $[-1, 1]$ **d)** $[1, \infty)$, $\emptyset$ **51. a)** $\{1, 2, 3, 4, 7, 8, 9, 10\}$ **b)** $\{2, 4, 5, 6, 7\}$ **c)** $\{1, 10\}$ **53.** The bit in the ith position of the bit string of the difference of two sets is 1 if the ith bit of the first string is 1 and the ith bit of the second string is 0, and is 0 otherwise. **55. a)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $=$ 11 1110 1000 0000 0100 0101 0000, representing $\{a, b, c, d, e, g, p, t, v\}$ **b)** 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 $=$ 01 1100 0000 0000 0000 0000 0000, representing $\{b, c, d\}$ **c)** (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$ (01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) $=$ 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 $=$ 01 1110 0010 0000 1000 0110 0110, representing $\{b, c, d, e, i, o, t, u, x, y\}$ **d)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 $=$ 11 1110 1110 0001 1100 0111 0111, representing $\{a, b, c, d, e, g, h, i, n, o, p, t, u, v, x, y, z\}$ **57. a)** $\{1, 2, 3, \{1, 2, 3\}\}$ **b)** $\{\emptyset\}$ **c)** $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ **59. a)** $\{3 \cdot a,\ 3 \cdot b,\ 1 \cdot c,\ 4 \cdot d\}$ **b)** $\{2 \cdot a,\ 2 \cdot b\}$ **c)** $\{1 \cdot a,\ 1 \cdot c\}$ **d)** $\{1 \cdot b,\ 4 \cdot d\}$ **e)** $\{5 \cdot a,\ 5 \cdot b,\ 1 \cdot c,\ 4 \cdot d\}$ **61.** $\overline{F} = \{0.4$ Alice, 0.1 Brian, 0.6 Fred, 0.9 Oscar, 0.5 Rita$\}$, $\overline{R} = \{0.6$ Alice, 0.2 Brian, 0.8 Fred, 0.1 Oscar, 0.3 Rita$\}$ **63.** $F \cap R = \{0.4$ Alice, 0.8 Brian, 0.2 Fred, 0.1 Oscar, 0.5 Rita$\}$

Section 2.3

1. a) $f(0)$ is not defined. **b)** $f(x)$ is not defined for $x < 0$. **c)** $f(x)$ is not well-defined because there are two distinct values assigned to each x. **3. a)** Not a function **b)** A function **c)** Not a function **5. a)** Domain the set of bit strings; range the set of integers **b)** Domain the set of bit strings; range the set of even nonnegative integers **c)** Domain the set of bit strings; range the set of nonnegative integers not exceeding 7 **d)** Domain the set of positive integers; range the set of squares of positive integers $= \{1, 4, 9, 16, \ldots\}$ **7. a)** Domain $\mathbf{Z}^+ \times \mathbf{Z}^+$; range $\mathbf{Z}^+$ **b)** Domain $\mathbf{Z}^+$; range $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **c)** Domain the set of bit strings; range $\mathbf{N}$ **d)** Domain the set of bit strings; range $\mathbf{N}$ **9. a)** 1 **b)** 0 **c)** 0 **d)** -1 **e)** 3 **f)** -1 **g)** 2 **h)** 1 **11.** Only the function in part (a) **13.** Only the functions in parts (a) and (d) **15. a)** Onto **b)** Not onto **c)** Onto **d)** Not onto **e)** Onto **17. a)** The function $f(x)$ with $f(x) = 3x + 1$ when $x \geq 0$ and $f(x) = -3x + 2$ when $x < 0$ **b)** $f(x) = |x| + 1$ **c)** The function $f(x)$ with $f(x) = 2x + 1$ when $x \geq 0$ and $f(x) = -2x$ when $x < 0$ **d)** $f(x) = x^2 + 1$ **19. a)** Yes **b)** No **c)** Yes **d)** No **21.** Suppose that f is strictly decreasing. This means that $f(x) > f(y)$ whenever $x < y$. To show that g is strictly increasing, suppose that $x < y$. Then $g(x) = 1/f(x) < 1/f(y) = g(y)$. Conversely, suppose that g is strictly increasing. This means that $g(x) < g(y)$ whenever $x < y$. To show that f is strictly decreasing, suppose that $x < y$. Then $f(x) = 1/g(x) > 1/g(y) = f(y)$. **23.** Many answers are possible. One example is $f(x) = 17$. **25.** The function is not one-to-one, so it is not invertible. On the restricted domain, the function is the identity function on the nonnegative real numbers, $f(x) = x$, so it is its own inverse. **27. a)** $f(S) = \{0, 1, 3\}$ **b)** $f(S) = \{0, 1, 3, 5, 8\}$ **c)** $f(S) = \{0, 8, 16, 40\}$ **d)** $f(S) = \{1, 12, 33, 65\}$ **29. a)** Let x and y be distinct elements of A. Because g is one-to-one, $g(x)$ and $g(y)$ are distinct elements of B. Because f is one-to-one, $f(g(x)) = (f \circ g)(x)$ and $f(g(y)) = (f \circ g)(y)$ are distinct elements of C. Hence, $f \circ g$ is one-to-one. **b)** Let $y \in C$. Because f is onto, $y = f(b)$ for some $b \in B$. Now because g is onto, $b = g(x)$ for some $x \in A$. Hence, $y = f(b) = f(g(x)) = (f \circ g)(x)$. It follows that $f \circ g$ is onto. **31.** No. For example, suppose that $A = \{a\}$, $B = \{b, c\}$, and $C = \{d\}$. Let $g(a) = b$, $f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not. **33.** $(f + g)(x) = x^2 + x + 3$, $(fg)(x) = x^3 + 2x^2 + x + 2$ **35.** f is one-to-one because $f(x_1) = f(x_2) \rightarrow ax_1 + b = ax_2 + b \rightarrow ax_1 = ax_2 \rightarrow x_1 = x_2$. f is onto because $f((y - b)/a) = y$. $f^{-1}(y) = (y - b)/a$. **37.** Let $f(1) = a$, $f(2) = a$. Let $S = \{1\}$ and $T = \{2\}$. Then $f(S \cap T) = f(\emptyset) = \emptyset$, but $f(S) \cap f(T) = \{a\} \cap \{a\} = \{a\}$. **39. a)** $\{x \mid 0 \leq x < 1\}$ **b)** $\{x \mid -1 \leq x < 2\}$ **c)** $\emptyset$ **41.** $f^{-1}(\overline{S}) = \{x \in A \mid f(x) \notin S\} = \overline{\{x \in A \mid f(x) \in S\}} = \overline{f^{-1}(S)}$ **43.** Let $x = \lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. If $\epsilon < \frac{1}{2}$, then $\lfloor x \rfloor - 1 < x - \frac{1}{2} < \lfloor x \rfloor$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$ and this is the integer closest to x. If $\epsilon > \frac{1}{2}$, then $\lfloor x \rfloor < x - \frac{1}{2} < \lfloor x \rfloor + 1$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor + 1$ and this is the integer closest to x. If $\epsilon = \frac{1}{2}$, then $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$, which is the smaller of the two integers that surround x and are the same distance from x. **45.** Write the real number x as $\lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. Because $\epsilon = x - \lfloor x \rfloor$, it follows that $0 \leq -\lfloor x \rfloor < 1$. The first two inequalities, $x - 1 < \lfloor x \rfloor$ and $\lfloor x \rfloor \leq x$, follow directly. For the other two inequalities, write $x = \lceil x \rceil - \epsilon'$, where $0 \leq \epsilon' < 1$. Then $0 \leq \lceil x \rceil - x < 1$, and the desired inequality follows. **47. a)** If $x < n$, because $\lfloor x \rfloor \leq x$, it follows that $\lfloor x \rfloor < n$. Suppose that $x \geq n$. By the definition of the floor function, it follows that $\lfloor x \rfloor \geq n$. This means that if $\lfloor x \rfloor < n$, then $x < n$. **b)** If $n < x$, then because $x \leq \lceil x \rceil$, it follows that $n \leq \lceil x \rceil$. Suppose that $n \geq x$. By the definition of the ceiling function, it follows that $\lceil x \rceil \leq n$. This means that if $n < \lceil x \rceil$, then $n < x$. **49.** If n is even, then $n = 2k$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k \rfloor = k = n/2$. If n is odd, then $n = 2k + 1$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k + \frac{1}{2} \rfloor = k = (n - 1)/2$. **51.** Assume that $x \geq 0$. The left-hand side is $\lceil -x \rceil$ and the right-hand side is $-\lfloor x \rfloor$. If x is an integer, then both sides equal $-x$. Otherwise, let $x = n + \epsilon$, where n is a natural number and ϵ is a real number with $0 \leq \epsilon < 1$. Then $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n$ and $-\lfloor x \rfloor = -\lfloor n + \epsilon \rfloor = -n$ also. When $x < 0$, the equation also holds because it can be obtained by substituting $-x$ for x. **53.** $\lceil b \rceil - \lfloor a \rfloor - 1$ **55. a)** 1 **b)** 3 **c)** 126 **d)** 3600 **57. a)** 100 **b)** 256 **c)** 1030 **d)** 30,200

59.



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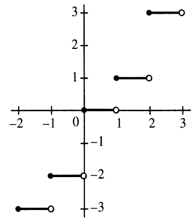
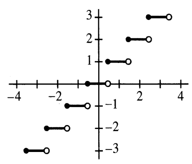
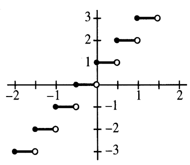


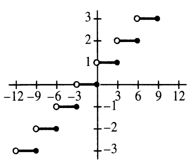
63. a)



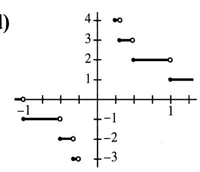
b)



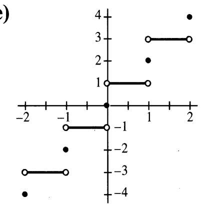
c)



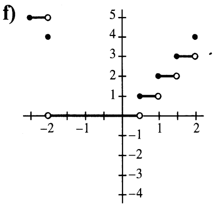
d)



e)



f)



g) See part (a). **65.** $f^{-1}(y) = (y - 1)^{1/3}$ **67. a)** $f_{A \cap B}(x) = 1 \leftrightarrow x \in A \cap B \leftrightarrow x \in A$ and $x \in B \leftrightarrow f_A(x) = 1$ and

$f_B(x) = 1 \leftrightarrow f_A(x)f_B(x) = 1$ **b)** $f_{A\cup B}(x) = 1 \leftrightarrow x \in A \cup B \leftrightarrow x \in A$ or $x \in B \leftrightarrow f_A(x) = 1$ or $f_B(x) = 1 \leftrightarrow f_A(x) + f_B(x) - f_A(x)f_B(x) = 1$ **c)** $f_{\overline{A}}(x) = 1 \leftrightarrow x \in \overline{A} \leftrightarrow x \notin A \leftrightarrow f_A(x) = 0 \leftrightarrow 1 - f_A(x) = 1$ **d)** $f_{A\oplus B}(x) = 1 \leftrightarrow x \in A \oplus B \leftrightarrow (x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B) \leftrightarrow f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1$ **69. a)** True; because $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. **b)** False; $x = \frac{1}{2}$ is a counterexample. **c)** True; if x or y is an integer, then by property 4b in Table 1, the difference is 0. If neither x nor y is an integer, then $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $m + n < x + y < m + n + 2$, so $\lceil x + y \rceil$ is either $m + n + 1$ or $m + n + 2$. Therefore, the given expression is either $(n + 1) + (m + 1) - (m + n + 1) = 1$ or $(n + 1) + (m + 1) - (m + n + 2) = 0$, as desired. **d)** False; $x = \frac{1}{4}$ and $y = 3$ is a counterexample. **e)** False; $x = \frac{1}{2}$ is a counterexample. **71. a)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m + \epsilon$, where n^2 is the largest perfect square less than x, m is a nonnegative integer, and $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n + 1$, so both sides equal n. **b)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 - m - \epsilon$, where n^2 is the smallest perfect square greater than x, m is a nonnegative integer, and ϵ is a real number with $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n - 1$ and n. Therefore, both sides of the equation equal n. **73. a)** Domain is $\mathbf{Z}$; codomain is $\mathbf{R}$; domain of definition is the set of nonzero integers; the set of values for which f is undefined is $\{0\}$; not a total function. **b)** Domain is $\mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definition is $\mathbf{Z}$; set of values for which f is undefined is $\emptyset$; total function. **c)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Q}$; domain of definition is $\mathbf{Z} \times (\mathbf{Z} - \{0\})$; set of values for which f is undefined is $\mathbf{Z} \times \{0\}$; not a total function. **d)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definition is $\mathbf{Z} \times \mathbf{Z}$; set of values for which f is undefined is $\emptyset$; total function. **e)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definitions is $\{(m, n) \mid m > n\}$; set of values for which f is undefined is $\{(m, n) \mid m \le n\}$; not a total function. **75. a)** By definition, to say that S has cardinality m is to say that S has exactly m distinct elements. Therefore we can assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence. **b)** By part (a), there is a bijection f from S to $\{1, 2, \ldots, m\}$ and a bijection g from T to $\{1, 2, \ldots, m\}$. Then the composition $g^{-1} \circ f$ is the desired bijection from S to T. **77.** It is clear from the formula that the range of values the function takes on for a fixed value of $m + n$, say $m + n = x$, is $(x - 2)(x - 1)/2 + 1$ through $(x - 2)(x - 1)/2 + (x - 1)$, because m can assume the values $1, 2, 3, \ldots, (x - 1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m + n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x + 1$ picks up precisely where the range of values for x left off, i.e., that $f(x - 1, 1) + 1 = f(1, x)$. We have $f(x - 1, 1) + 1 = \frac{(x-2)(x-1)}{2} + (x - 1) + 1 = \frac{x^2 - x + 2}{2} = \frac{(x-1)x}{2} + 1 = f(1, x)$.

Section 2.4

1. a) 3 **b)** -1 **c)** 787 **d)** 2639 **3. a)** $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$ **b)** $a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256$ **c)** $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$ **d)** $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$ **5. a)** 2, 5, 8, 11, 14, 17, 20, 23, 26, 29 **b)** 1, 1, 1, 2, 2, 2, 3, 3, 3, 4 **c)** 1, 1, 3, 3, 5, 5, 7, 7, 9, 9 **d)** $-1, -2, -2$, 8, 88, 656, 4912, 40064, 362368, 3627776 **e)** 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536 **f)** 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 **g)** 1, 2, 2, 3, 3, 3, 3, 4, 4, 4 **h)** 3, 3, 5, 4, 4, 3, 5, 5, 4, 3 **7.** Each term could be twice the previous term; the nth term could be obtained from the previous term by adding $n - 1$; the terms could be the positive integers that are not multiples of 3; there are infinitely many other possibilities. **9. a)** One 1 and one 0, followed by two 1s and two 0s, followed by three 1s and three 0s, and so on; 1, 1, 1 **b)** The positive integers are listed in increasing order with each even positive integer listed twice; 9, 10, 10. **c)** The terms in odd-numbered locations are the successive powers of 2; the terms in even-numbered locations are all 0; 32, 0, 64. **d)** $a_n = 3 \cdot 2^{n-1}$; 384, 768, 1536 **e)** $a_n = 15 - 7(n - 1) = 22 - 7n$; $-34, -41, -48$ **f)** $a_n = (n^2 + n + 4)/2$; 57, 68, 80 **g)** $a_n = 2n^3$; 1024, 1458, 2000 **h)** $a_n = n! + 1$; 362881, 3628801, 39916801 **11.** Among the integers $1, 2, \ldots, a_n$, where a_n is the nth positive integer not a perfect square, the nonsquares are $a_1, a_2, \ldots, a_n$ and the squares are $1^2, 2^2, \ldots, k^2$, where k is the integer with $k^2 < n + k < (k + 1)^2$. Consequently, $a_n = n + k$, where $k^2 < a_n < (k + 1)^2$. To find k, first note that $k^2 < n + k < (k + 1)^2$, so $k^2 + 1 \le n + k \le (k + 1)^2 - 1$. Hence, $(k - \frac{1}{2})^2 + \frac{3}{4} = k^2 - k + 1 \le n \le k^2 + k = (k + \frac{1}{2})^2 - \frac{1}{4}$. It follows that $k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2}$, so $k = \{\sqrt{n}\}$ and $a_n = n + k = n + \{\sqrt{n}\}$. **13. a)** 20 **b)** 11 **c)** 30 **d)** 511 **15. a)** 1533 **b)** 510 **c)** 4923 **d)** 9842 **17. a)** 21 **b)** 78 **c)** 18 **d)** 18 **19.** $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$ **21. a)** n^2 **b)** $n(n + 1)/2$ **23.** 15150 **25.** $\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} + (n + 1)(m - (n + 1)^2 + 1)$, where $n = \lfloor \sqrt{m} \rfloor - 1$ **27. a)** 0 **b)** 1680 **c)** 1 **d)** 1024 **29.** 34 **31. a)** Countable, $-1, -2, -3, -4, \ldots$ **b)** Countable, $0, 2, -2, 4, -4, \ldots$ **c)** Uncountable **d)** Countable, $0, 7, -7, 14, -14, \ldots$ **33. a)** Countable: match n with the string of n 1s. **b)** Countable. To find a correspondence, follow the path in Example 20, but omit fractions in the top three rows (as well as continuing to omit fractions not in lowest terms). **c)** Uncountable **d)** Uncountable **35.** Assume that $A - B$ is countable. Then, because $A = (A - B) \cup B$, the elements of A can be listed in a sequence by alternating elements of $A - B$ and elements of B. This contradicts the uncountability of A. **37.** Assume that B is countable. Then the elements of B can be listed as $b_1, b_2, b_3, \ldots$. Because A is a subset of B, taking the subsequence of $\{b_n\}$ that contains the terms that are in A gives a listing of the elements of A. Because A is uncountable, this is impossible. **39.** We are given bijections f from A to B and g from C to D. Then the function from $A \times C$ to $B \times D$ that sends (a, c) to $(f(a), g(c))$ is a bijection. **41.** Suppose that $A_1, A_2, A_3, \ldots$ are countable sets. Because A_i is countable, we can list its elements in a sequence as $a_{i1}, a_{i2}, a_{i3}, \ldots$. The elements of the set $\bigcup_{i=1}^{n} A_i$ can be listed by listing all terms a_{ij} with $i + j = 2$, then all terms a_{ij} with $i + j = 3$, then all terms a_{ij} with $i + j = 4$, and so on. **43.** There are a finite number of bit strings of length m, namely, 2^m. The set of all bit, strings is the union of the sets of bit strings of length m for $m = 0, 1, 2, \ldots$. Because the union of a countable number of countable sets is countable (see Exercise 41), there are a countable number of bit strings. **45.** For any finite alphabet there are a finite number of strings of length n, whenever n is a positive integer. It follows by the result of Exercise 41 that there are only a countable number of strings from any given finite alphabet. Because the set of all computer programs in a particular language is a subset of the set of all strings of a finite alphabet, which is a countable set by the result from Exercise 36, it is itself a countable set. **47.** Exercise 45 shows that there are only a countable number of computer programs. Consequently, there are only a countable number of computable functions. Because, as Exercise 46 shows, there are an uncountable number of functions, not all functions are computable.

Supplementary Exercises

1. a) $\overline{A}$ **b)** $A \cap B$ **c)** $A - B$ **d)** $\overline{A} \cap \overline{B}$ **e)** $A \oplus B$ **3.** Yes **5.** $A-(A-B) = A-(A\cap\overline{B}) = A\cap\overline{(A\cap\overline{B})} = A \cap (\overline{A} \cup B) = (A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$ **7.** Let $A = \{1\}, B = \emptyset, C = \{1\}$. Then $(A - B) - C = \emptyset$, but $A - (B - C) = \{1\}$. **9.** No. For example, let $A = B = \{a, b\}$, $C = \emptyset$, and $D = \{a\}$. Then $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, but $(A - C) - (B - D) = \{a, b\} - \{b\} = \{a\}$. **11. a)** $|\emptyset| \le |A \cap B| \le |A| \le |A \cup B| \le |U|$ **b)** $|\emptyset| \le |A - B| \le |A \oplus B| \le |A \cup B| \le |A| + |B|$ **13. a)** Yes, no **b)** Yes, no **c)** f has inverse with $f^{-1}(a) = 3, f^{-1}(b) = 4, f^{-1}(c) = 2, f^{-1}(d) = 1$; g has no inverse. **15.** Let $f(a) = f(b) = 1$, $f(c) = f(d) = 2$, $S = \{a, c\}$, $T = \{b, d\}$. Then $f(S \cap T) = f(\emptyset) = \emptyset$, but $f(S) \cap f(T) = \{1, 2\} \cap \{1, 2\} = \{1, 2\}$. **17.** Let $x \in A$. Then $S_f(\{x\}) = \{f(y) \mid y \in \{x\}\} = \{f(x)\}$. By the same reasoning, $S_g(\{x\}) = \{g(x)\}$. Because $S_f = S_g$, we can conclude that $\{f(x)\} = \{g(x)\}$, and so necessarily $f(x) = g(x)$. **19.** The equation is true if and only if the sum of the fractional parts of x and y is less than 1. **21.** The equation is true if and only if either both x and y are integers, or x is not an integer but the sum of the fractional parts of x and y is less than or equal to 1. **23.** If x is an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = x + m - x = m$. Otherwise, write x in terms of its integer and fractional parts: $x = n + \epsilon$, where $n = \lfloor x \rfloor$ and $0 < \epsilon < 1$. In this case $\lfloor x \rfloor + \lfloor m - x \rfloor = \lfloor n + \epsilon \rfloor + \lfloor m - n - \epsilon \rfloor = n + m - n - 1 = m - 1$. **25.** Write $n = 2k + 1$ for some integer k. Then $n^2 = 4k^2 + 4k + 1$, so $n^2/4 = k^2 + k + \frac{1}{4}$. Therefore, $\lceil n^2/4 \rceil = k^2 + k + 1$. But $(n^2 + 3)/4 = (4k^2 + 4k + 1 + 3)/4 = k^2 + k + 1$. **27.** Let $x = n + (r/m) + \epsilon$, where n is an integer, r is a nonnegative integer less than m, and ϵ is a real number with $0 \le \epsilon < 1/m$. The left-hand side is $\lfloor nm + r + m\epsilon \rfloor = nm + r$. On the right-hand side, the terms $\lfloor x \rfloor$ through $\lfloor x + (m + r - 1)/m \rfloor$ are all just n and the terms from $\lfloor x + (m - r)/m \rfloor$ on are all $n + 1$. Therefore, the right-hand side is $(m - r)n + r(n + 1) = nm + r$, as well. **29.** 101 **31.** $a_1 = 1$; $a_{2n+1} = n \cdot a_{2n}$ for all $n > 0$; and $a_{2n} = n + a_{2n-1}$ for all $n > 0$. The next four terms are 5346, 5353, 37471, and 37479.

CHAPTER 3

Section 3.1

1. $max := 1$, $i := 2$, $max := 8$, $i := 3$, $max := 12$, $i := 4$, $i := 5$, $i := 6$, $i := 7$, $max := 14$, $i := 8$, $i := 9$, $i := 10$, $i := 11$

3.
```
procedure sum(a_1, ..., a_n: integers)
  sum := a_1
  for i := 2 to n
    sum := sum + a_i
  {sum has desired value}
```

5.
```
procedure duplicates(a_1, a_2, ..., a_n: integers in
    nondecreasing order)
  k := 0 {this counts the duplicates}
  j := 2
  while j ≤ n
  begin
    if a_j = a_{j-1} then
    begin
      k := k + 1
      c_k := a_j
      while (j ≤ n and a_j = c_k)
        j := j + 1
    end
    j := j + 1
  end {c_1, c_2, ..., c_k is the desired list}
```

7.
```
procedure last even location(a_1,a_2, ...,a_n: integers)
  k := 0
  for i := 1 to n
    if a_i is even then k := i
  end {k is the desired location (or 0 if there are no evens)}
```

9.
```
procedure palindrome check(a_1a_2 ... a_n: string)
  answer := true
  for i := 1 to ⌊n/2⌋
    if a_i ≠ a_{n+1-i} then answer := false
  end {answer is true iff string is a palindrome}
```

11.
```
procedure interchange(x, y: real numbers)
  z := x
  x := y
  y := z
```
The minimum number of assignments needed is three.

13. Linear search: $i := 1, i := 2, i := 3, i := 4, i := 5, i := 6$, $i := 7$, $location := 7$; binary search: $i := 1$, $j := 8$, $m := 4$, $i := 5$, $m := 6$, $i := 7$, $m := 7$, $j := 7$, $location := 7$

15.
```
procedure insert(x, a_1, a_2, ..., a_n: integers)
  {the list is in order: a_1 ≤ a_2 ≤ ··· ≤ a_n}
  a_{n+1} := x + 1
  i := 1
```